Partial Differential Equations - Final exam

You have 2 hours to complete this exam. Please show all work. 10 points for writing your name and student id on every page. Be sure to quote clearly any theorems you use from the textbook or class. Good luck!

(1) (25 points) Let $[0, \ell]$ be an interval of length ℓ. Consider the equation

$$\begin{aligned} &\partial_t u(t,x) = k\partial_x^2 u(t,x) \qquad \text{in} \quad [0,\ell], \quad t,k>0 \\ &u(0,x) = f(x) \\ &u(t,\ell) = 20 \quad u(t,0) = 10 \end{aligned}$$

(a) (12 points) Solve the equation if $f(x) = \sin(\frac{3\pi x}{\ell})$.

(b) (5 points) What is the maximum value of $u(t,x)$ on $[0,\ell]$ with $f(x)$ as in (a) and why?

(c) (8 points) Assume instead that $f(x) \in L^1([0,\ell])$ that is, $\int_0^\ell |f(x)|\,dx < \infty$. What is the value of $|u(t,x)|$ as $t \to \infty$?

(2) (25 points) Consider the initial value problem

$$\begin{cases} -u'' + \sigma^2 u = h(x), \quad x \in \mathbb{R},\ \sigma > 0, \\ u(x) \to 0 \text{ as } |x| \to \infty \end{cases} \tag{0.1}$$

(a) (5 points) Give the definition of the free space Green's function for the Poisson problem.
(b) (10 points) Show that, if you know the free space Green's function G_0, you can compute the solution of (0.1).
(c) (10 points) Use the Fourier Transform to directly solve (0.1) in terms of a convolution.

(3) (20 points total) We have the following second order partial differential equation

$$4u_{xx} + 6u_{xy} + 9u_{yy} = 0 \tag{0.2}$$

(a) (5 points) Classify the partial differential equation.
(b) (7 points) Find the canonical form.
(c) (8 points) Find the change of variables which brings the solution of the canonical form back to the original coordinate system.

(4) (20 points total) Consider the Cauchy problem for the wave equation

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad x \in \mathbb{R}, t > 0, c \in \mathbb{R} \\ &u(0,x) = f(x) \\ &\frac{\partial u}{\partial t}(0,x) = g(x) \end{aligned}$$

(a) (10 points) Use the energy method to show that for $f, g \in C_0^2(\mathbb{R})$ that the solution to this equation is unique in $C_0^2(\mathbb{R})$. Use the energy function

$$E(t) = \int_{\mathbb{R}} c^2 |\partial_x u|^2 + |\partial_t u|^2 \, dx \tag{0.3}$$

in your solution.

(b) (7 points) Solve the problem using D'Alembert's principle with $g(x) = -2xe^{-x^2}$ and $f(x) = e^{-x^2}$.

(c) (3 points) What do you know about the solution of the problem with $g(x) = -2xe^{-x^2}$ and $f(x) = e^{-x^2}$ when solved using the Fourier transform? (hint, use part a).